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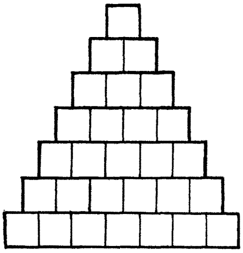
THE FORMULA $\frac{1}{2}a(a+1)$ FOR THE AREA OF AN EQUILATERAL TRIANGLE.

By G. A. MILLER, University of Illinois.

Among the interesting facts connected with the formula $\frac{1}{2}a(a + 1)$ for the area of an equilateral triangle whose side is a are the following: It is remarkably inaccurate, the figures relating thereto which appear in various well known histories of mathematics are misleading, and an attempt to explain why it fails to yield correct results has been called "the first mathematical paper of the Middle Ages which deserves this name."

As regards the first of these points it may be sufficient to direct attention to the fact that in the work of Ahmes, written about 1700 B.C., the area of an isosceles triangle seems to have been found by multiplying one-half the base by one of the equal sides instead of by the altitude. This fact has been regarded as noteworthy, but when it is observed that more than two thousand years later Roman surveyors were taught to calculate the area of a special isosceles triangle by multiplying the numerical measure of half the base by a number which is even larger than the numerical measure of another side there seems to be sufficient ground for being surprised.

The present note does not aim to give a sketch of the history of this interesting formula, which sheds much light on the lack of geometric insight of the Roman surveyors during the first few centuries of the Middle Ages, appearing in such writings as Boethius's Geometry and Gerbert's Geometry. The main object of this note is to supplement statements and figures relating to this formula which appear in some of our best known modern histories of mathematics. The full benefit of the note can be secured only by those who consult the statements and figures to which references will be furnished but an effort has been made to make it instructive also to others.



The mathematics connected with the formula $\frac{1}{2}a(a + 1)$ for the area of an equilateral triangle whose side is a is very elementary. If a given line segment is divided into a equal parts and if squares are arranged thereon as in the adjoining figure it is obvious that the number of these squares is the triangular number $\frac{1}{2}a(a + 1)$, and that as a increases the totality of these squares approaches the area of an isosceles triangle whose altitude is equal to the base. Hence it results that if the formula $\frac{1}{2}a(a + 1)$ were used to calculate the area of a given isosceles triangle whose altitude and base are both equal to a it would represent this area more and more closely as the unit of measure of the base is diminished, and the approximation could thus be made arbitrarily close by taking the unit of measure sufficiently small.

On the other hand, when the area of an equilateral triangle whose side is a

is computed by this same formula the area thus found will approach a limit, as a increases, whose ratio is to the true area as 2 is to $\sqrt{3}$. That is, the error in this case is always larger than 15 per cent.

The reason for referring here to these obvious facts is that Gerbert, who died in 1003 as Pope Sylvester II, attempted to explain in a letter to Adelbold why the area of an equilateral triangle whose side is 7 when computed by the formula $\frac{1}{2}a(a+1)$ was too large. He based his arguments on the fact that squares lying only partly within the equilateral triangle were counted according to this formula as if they were entirely within the triangle. The correctness of this part of the explanation in this particular case can be established by consulting the given figure since an equilateral triangle on the base of this figure has its vertex very slightly above the base of the uppermost square.

One of the objects of the present note is to direct attention to the fact that in the three editions of volume I of Cantor's *Vorlesungen über Geschichte der Mathematik*, 1880, 1894, 1907, pages 744, 815 and 866 respectively, the figure relating to this triangle is inaccurate. What is still more important is the fact that the corresponding figure found in various other histories is still more misleading since it represents according to the explanations in the text an isosceles triangle whose base is equal to the altitude while the text itself relates to an equilateral triangle. This fact can be verified by consulting either edition of Cajori's *History of Elementary Mathematics,* 1896 or 1917, page 132, or Günther's *Geschichte der Mathematik,* 1908, page 249. Unless the inaccuracy of these figures is noted the reader is apt to draw incorrect conclusions in regard to the merits of Gerbert's attempted explanation, which was called inaccurate by M. Chasles in his well known *Aperçu Historique,* 1875, page 506, but has been called correct in each of the mathematical histories to which reference was made above.

In view of the fact that we are now living in an age of numerous mathematical papers it may be of special interest to note here that H. Hankel[1] called Gerbert's letter to which we referred "the first mathematical paper of the Middle Ages which deserves this name." The interest which such an assertion awakens is reflected in the fact that F. Cajori quotes this statement in both editions of his *History of Mathematics,* 1894 and 1919, as well as in both editions of his *History of Elementary Mathematics,* 1896 and 1917. This high epithet of Gerbert's letter is based on the attempted explanation to which we referred and hence the appropriateness of this epithet is necessarily called in question if this explanation is regarded as inaccurate or trivial.

The triviality of the explanation results directly from the figure given above since it is obvious that the uppermost square lies above the vertex of the equilateral triangle on the same base whenever the integer $a > 7$. This triviality becomes the more obvious if it is noted that in the first part of the letter in question Gerbert speaks of a triangle whose side is 30 and his explanation would fail entirely in this case. If this triviality is granted it follows directly that the letter in question does not merit the epithet "the first mathematical paper of the Middle Ages which deserves this name."

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[1] In his *Geschichte der Mathematik,* 1874, p. 314.

The present note deals admittedly with questions whose satisfactory treatment would require a large amount of space. It is hoped, however, that a few points have become perfectly clear and that these may throw light on others. In the first place, there seems to be no room for doubt in regard to the inaccuracy of the figures appearing in various well and favorably known histories of mathematics which aim to present Gerbert's explanation of the formula $\frac{1}{2}a(a+1)$ for the area of an equilateral triangle whose side is a. A correct figure appears in Bubnov's *Gerberti Opera Mathematica,* 1899, Tab. I. In the second place, it should be clear that Gerbert's attempted explanation of the formula in question exhibits too little mathematical insight and is too trivial to merit the epithet which has been so widely attributed to it. While the time from the beginning of the Middle Ages to the end of the tenth century does not exhibit much creditable work in mathematics it does present some noteworthy advances, especially in algebra and trigonometry.

NOTE ON THE PRIME DIVISORS OF THE NUMERATORS OF BERNOULLI'S NUMBERS.

By E. T. BELL, University of Washington.

1. Using the even-suffix notation for Bernoulli's numbers, $B_0 = 1$, $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$, $\cdots$ $B_{12} = -691/2730$, $B_{14} = 7/6$, $B_{16} = -3617/510$, $\cdots$, as in Lucas, *Théorie des Nombres,* Chap. XIV, we shall prove the following

THEOREM. *If p is an odd prime which does not divide $4^r - 1$, the numerator of B_{2pr} is divisible by p.*

Hence for $r = 1$ we have a result due to John Couch Adams:[1]

COROLLARY. *If $p > 3$ is a prime, the numerator of B_{2p} is divisible by p.*

Both of these are useful as checks in numerical work, also they have a certain theoretical interest in some parts of arithmetic. Another observation due to Adams (quoted by Lucas, p. 435), states[2] that if p is an odd prime divisor of q and not a divisor of the denominator of B_{2q}, the numerator of B_{2q} is divisible by p. A comparison of this result and that which we shall establish shows that in numerical work one can often be applied with less labor than the other.

2. The proof depends upon the known fact that for $q > 0$ an integer,

$$I_q \equiv 2^{2q-1}(2^{2q} - 1)B_{2q}/q$$

is an integer.[3] Assume this for a moment; write $q = pr$, where p is an odd prime, and put $B_{2q} = N_{2q}/D_{2q}$, N_{2q}, D_{2q} being the numerator and denominator respectively of B_{2q}. Then

$$I_{pr} = 2^{2pr-1}(2^{2pr} - 1)N_{2pr}/prD_{2pr}.$$

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[1] *Scientific Papers of John Couch Adams,* vol. 1, 1896, pp. xliv, 430.—EDITOR.

[2] J. C. Adams, Crelle's *Journal,* vol. 85, 1878, p. 269; *Scientific Papers,* vol. 1, p. 430.—EDITOR.

[3] Compare *Encyklopädie der Mathematischen Wissenschaften,* vol. II–1, 2–3, 1899, p. 183.—EDITOR.